

# Folded States of Single Vertex Origami And Miura-Ori

Roger Alperin and Jason Orozco

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## Abstract

We introduce an easily computable equivalence relation on crease patterns giving only finitely many equivalence classes, called  $K$ -points, for a single vertex (SV) with  $2n$  angles, and also similarly for a Miura-ori (MO) with  $m$  rows and  $n$  columns.

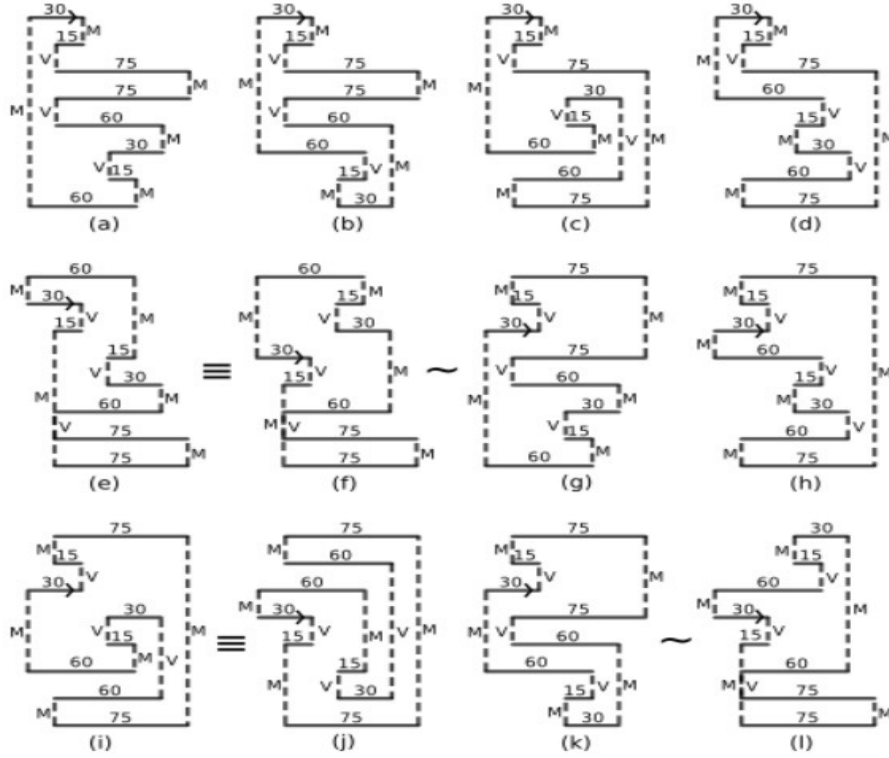
Two SV crease patterns which are equivalent have exactly the same flat-folded states (FFS). For  $n = 2$  any SV crease pattern is equivalent to a) all  $90^\circ$ ; b) two sets of equal angles e. g.  $60^\circ, 60^\circ, 120^\circ, 120^\circ$ ; c) one set of equal angles e.g.  $90^\circ, 45^\circ, 90^\circ, 135^\circ$ ;  $K_2 = 3$ . A table of the 29 possible  $K$ -point equivalence classes for  $n = 3$ , ie  $K_3 = 29$ , is shown on the right. In addition we get  $K_4 = 704, K_5 = 40251, K_6 = 36884596$ .

To determine the FFS of a SV we use closed meanders and the Justin conditions to limit the possibilities. For example with angles  $30^\circ, 15^\circ, 75^\circ, 75^\circ, 60^\circ, 30^\circ, 15^\circ, 60^\circ$  there are 10 different FFS and 8 mountain-valley (MV) assignments obtained using a possible 12 different closed meanders (M). These are shown in Figure 1 below; equivalent FFS are denoted by  $\equiv$ , and equal MV are denoted by  $\sim$ .

In the case of MO we develop an equivalence relation on the set of  $m$  of rows,  $n$  columns crease patterns using the acute angle  $\alpha$  of the base parallelogram and side length ratio  $\rho$ . We reduce to a finite set of equivalence classes using the ‘compression’ parameter  $c = \min\{\lceil \frac{\rho}{2\cos\alpha} \rceil, \lceil \frac{n}{2} \rceil\}$ . For a fixed  $m, n$  the FFS increases as the compression increases since there are more rearrangement of flaps taking place, but the number of MV assignments decreases.

In the table below labelled 3x6 MO compressions, we list the FFS and MV assignments for the MO with 3 rows and 6 columns with  $\alpha = 45^\circ$  and  $\rho = 1, 2, 3$  giving compressions  $c = 1, 2, 3$ .

K-Pt	FFS	MV
(1,1,1,1,1,1)	24	15
(1,1,1,1,2,2)	14	10
(1,1,2,2,2,2)	15	12
(1,1,2,2,3,3)	11	9
(1,1,2,1,1,2)	9	9
(1,1,2,1,2,3)	6	6
(1,1,3,1,1,3)	9	9
(1,1,3,1,2,4)	6	6
(2,2,2,1,2,3)	8	6
(2,2,3,2,2,3)	12	9
(2,2,3,2,3,4)	8	6
(2,1,1,2,3,3)	9	9
(2,1,1,1,2,3)	8	6
(2,1,2,3,3,3)	8	8
(2,1,2,3,4,4)	6	6
(2,1,2,2,1,2)	6	4
(2,1,2,2,3,4)	6	4
(2,1,2,1,2,4)	4	4
(2,1,3,4,3,3)	6	6
(2,1,3,3,2,3)	5	4
(2,1,3,2,1,3)	4	4
(2,1,3,1,2,5)	4	4
(2,1,4,2,1,4)	4	4
(3,2,1,1,3,4)	6	6
(3,2,3,2,3,5)	6	4
(3,2,4,3,2,4)	5	4
(3,1,2,3,4,5)	4	4
(3,1,3,4,3,4)	4	4
(4,2,1,2,4,5)	4	4



**Figure 1:** 12M-10FFS-8MV

	$c$	1	2	3
3 x 6 MO compressions	FFS	6709	20996	27274
	MV	3278	3118	3101

We can also use our algorithms to compare the number of MV assignments with an upper bound (calculated in terms of colorings). The cases of  $(m, n) = (3, 3), (5, 2)$  have one coloring which does not come from a flat folding; in the case  $(5, 8)$  there are 89701014 colorings or local foldings which are not flat-foldable. Below is the table labelled MV showing half the number of MV assignments for  $c = 1$  in an MO with specified rows and columns matching those dimensions in the table of coloring counts (See J. Ginepro and T. Hull, Counting Miura-ori Foldings, J. of Integer Sequences, v. 17, 2014, 14.10.8).

	$m/n$	2	3	4	5	6	7	8
MV	2	3	9	27	81	243	729	2187
	3	9	40	175	759	3278	14129	60843
	4	27	171	1050	6367	38421	231418	1392903
	5	80	708	5994	49955	413982	3423778	28296029